

# Matter-wave analog of an optical random laser

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The accumulation of atoms in the lowest energy level of a trap and the subsequent out-coupling of these atoms is a realization of a matter-wave analog of a conventional optical laser. Optical random lasers require materials that provide optical gain but, contrary to conventional lasers, the modes are determined by multiple scattering and not a cavity. We show that a Bose-Einstein condensate can be loaded in a spatially correlated disorder potential prepared in such a way that the Anderson localization phenomenon operates as a band-pass filter. A multiple scattering process selects atoms with certain momenta and determines laser modes which represents a matter-wave analog of an optical random laser.

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Conventional optical lasers require two ingredients: material that provides optical gain and an optical cavity responsible for coherent feedback and selection of resonant laser modes. However, it is also possible to achieve laser action without the optical cavity provided the gain material is an active medium with disorder [1]. Forty years ago Letokhov analyzed a light diffusion process with amplification and predicted that gain could overcome loss if the volume of a system exceeded a critical value [2]. Random lasing (i.e. light amplification in disordered gain media), achieved in a laboratory in the 1990's, attracts much experimental attention and offers possibilities for interesting applications [1, 3–7]. Theoretical understanding of this phenomenon is still imperfect. Although the Letokhov model of diffusion with gain is useful in predicting certain properties of random lasers, it neglects coherent phenomena. There are various theoretical models of random lasing but it is widely accepted that interference in a multiple scattering process determines the spatial and spectral mode structure of a random laser [1].

Bose-Einstein condensation (BEC) of dilute atomic gases is a macroscopic accumulation of atoms in the lowest energy level of a trap when the temperature of the gas decreases [8]. This tendency of occupying a single state through the mechanism of stimulated scattering of bosons is an analog of mode selection in optical lasers due to the stimulated emission of photons. Gradual release of atoms from a trapped BEC allows for the realization of a matter-wave analog of a conventional optical laser [9–14]. The atom trap is an analog of the optical cavity. The lowest mode of the trap is a counterpart of an optical resonant mode. In conventional optical lasers the output coupler is usually a partially transmitting mirror. In atom lasers it involves, for example, a change of the internal state of the atom by means of a radio-frequency transition.

In the present letter we propose the realization of a matter-wave analog of an optical random laser. Suppose

the BEC of a dilute atomic gas has been achieved in a trapping potential. That is, we begin with the accumulation of atoms in a single mode of the resonator (i.e. the lowest eigenstate of the trap). Then, let us turn off the trap and turn on a weak disorder potential. Starting with a BEC we have a guarantee that the disorder medium is *pumped* with coherent matter-waves. We would like to raise the question of whether it is possible to prepare spatially correlated disorder potential in such a way that narrow peaks can be observed in the spectrum of atoms that are able to leave the area of the disorder potential? In other words: if the multiple scattering of atoms in a disorder medium can lead to a selective spectral emission of matter-waves from the medium?

In cold atom physics a disorder potential can be realized by means of an optical speckle potential [15, 16]. Transmission of coherent light through a diffusing plate leads to a random intensity pattern in the far field. Atoms experience the presence of the radiation as an external potential  $V(\mathbf{r}) \propto \chi |E(\mathbf{r})|^2$  proportional to the intensity of the light field  $E(\mathbf{r})$  and atomic polarizability  $\chi$  whose sign depends on the detuning of the light frequency from the atomic resonance. Diffraction from the diffusive plate onto the location of atoms determines correlation functions of the speckle potential. We assume that the origin of the energy is shifted so that  $\overline{V(\mathbf{r})} = 0$  where the overbar denotes an ensemble average over disorder realizations. Standard deviation of the speckle potential  $V_0$  measures the strength of the disorder.

Let us begin with a one-dimensional (1D) problem. In a weak disorder potential atoms with  $k$ -momentum undergo multiple scattering, diffusive motion and finally localize with an exponentially decaying density profile due to the Anderson localization process, provided that the system size exceeds the localization length [17–19]. Taking the Born approximation to the second order in the potential strength, the inverse of the localization length is [20]  $l_{loc}^{-1} = (mV_0/\hbar^2 k)^2 \mathcal{P}(2k)$ , where the Fourier transform of the pair correlation function of the speckle po-

tential is

$$\mathcal{P}(k) = \int \frac{dq}{2\pi} \gamma(q) \gamma(k-q), \quad (1)$$

and  $\gamma(k) = \int dz \tilde{\gamma}(z) e^{-ikz}$  is the Fourier transform of the complex degree of coherence  $\tilde{\gamma}(z) = \frac{E^*(z+z')E(z')}{|E(z)|^2} = \int dy \mathcal{A}(y) e^{izy/\alpha} / \int dy \mathcal{A}(y)$ .  $\mathcal{A}(y)$  describes the aperture of the optics and  $\alpha$  is a constant dependant on the wavelength of the laser radiation and the distance of the diffusive plate from the atomic trap. In Ref. [21], where the experimental realization of the Anderson localization of matter-waves is reported (see also [22]), a simple Heaviside step function  $\mathcal{A}(z) = \Theta(R - |z|)$  describes the aperture. The corresponding  $\gamma(k) = \pi \sigma_R \Theta(1 - |k\sigma_R|)$ , where  $\sigma_R = \alpha/R$  is the correlation length of the speckle potential. Consequently the power spectrum (1) decreases linearly and becomes zero for  $|k| \geq 2/\sigma_R$ . Thus, the Born approximation predicts an effective mobility edge at  $|k| = 1/\sigma_R$ , i.e. atoms with larger momenta do not localize [21, 23] (actually higher order calculations [24] show they do localize but with very large localization lengths, much larger than the system size in the experiment). Hence, neglecting atom interactions, if the width of the initial atom momentum distribution exceeds the mobility edge particles at the tail of the distribution avoid the Anderson localization and may leave the disorder area.

Let us modify the experiment reported in Ref. [21] by introducing an obstacle at the center of the diffusive plate so that the aperture is now described by  $\mathcal{A}(z) = \Theta(R - |z|) - \Theta(\rho - |z|)$  where  $\rho < R$ . It implies that

$$\gamma(k) = \pi \left( \frac{1}{\sigma_R} - \frac{1}{\sigma_\rho} \right)^{-1} [\Theta(1 - |k\sigma_R|) - \Theta(1 - |k\sigma_\rho|)], \quad (2)$$

where  $\sigma_\rho = \alpha/\rho$ . If the size of the obstacle  $\rho > R/3$ , interference of light passing through such a *double-slit* diffusive plate creates a peculiar speckle potential. That is, the power spectrum (1) disappears for  $|k| \geq 2/\sigma_R$  as previously but it is also zero for  $\frac{1}{\sigma_R} - \frac{1}{\sigma_\rho} < |k| < \frac{2}{\sigma_\rho}$ . Thus, according to the Born approximation there is a momentum interval where the localization length diverges. It implies that the Anderson localization process is able to operate as a band-pass filter letting particles with specific momenta leave the region of the disorder. Detection of escaping atoms should reveal a peak in the momentum spectrum corresponding to the interval where the localization length diverges.

Introducing two (or more) obstacles in the diffusive plate we can increase the number of momentum intervals with diverging  $l_{loc}$ . In Fig. 1 we present examples of the speckle potentials in the single obstacle case and the case of two obstacles located symmetrically around the plate center. In the latter case the aperture is described by  $\mathcal{A}(z) = \Theta(R - |z|) - \Theta(\rho - |z|) + \Theta(\zeta - |z|)$  where  $\zeta < \rho$

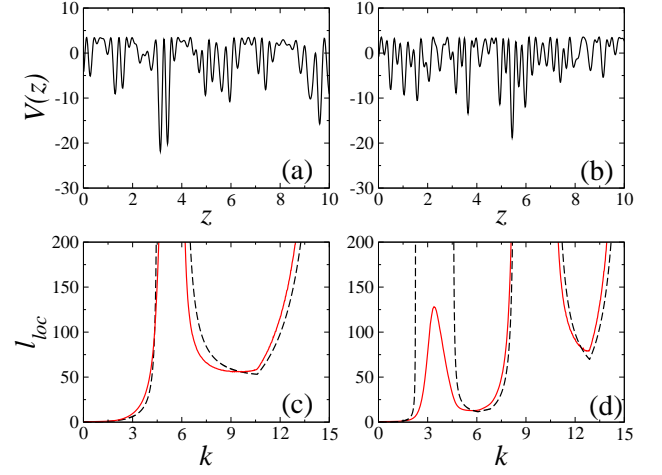


Figure 1: (Color online) Examples of the speckle potential (top panels) and the corresponding localization length (bottom panels) obtained within the Born approximation (dashed black curves) and numerically in the transfer-matrix calculations (solid red curves). Panels (a) and (c) show the results for the single obstacle in the diffusive plate where  $\sigma_R = 0.066$  ( $0.31 \mu\text{m}$ ),  $\sigma_R/\sigma_\rho = 0.4$  and  $V_0 = 3.5$ . Panels (b) and (d) correspond to the case of two obstacles with the same  $\sigma_R$  and  $V_0$  but  $\sigma_R/\sigma_\rho = 0.7$  and  $\sigma_R/\sigma_\zeta = 0.1$ . The results are shown for rubidium-87 atoms. Red detuning of the laser radiation from the atomic resonance is assumed. All values are presented in the harmonic oscillator units, i.e. energy  $E_0 = \hbar\omega$  and length  $l_0 = \sqrt{\hbar/m\omega}$  where  $\omega/2\pi = 5.4 \text{ Hz}$ .

and

$$\gamma(k) = \pi \left( \frac{1}{\sigma_R} - \frac{1}{\sigma_\rho} + \frac{1}{\sigma_\zeta} \right)^{-1} [\Theta(1 - |k\sigma_R|) - \Theta(1 - |k\sigma_\rho|) + \Theta(1 - |k\sigma_\zeta|)], \quad (3)$$

with  $\sigma_\zeta = \alpha/\zeta$ . The figure also presents localization lengths obtained numerically in the transfer-matrix calculation [24] that confirms the Born predictions.

To simulate an experiment, we follow the parameters used in Ref. [21] where Anderson localization of matter-waves has been observed. We assume that a BEC of  $N = 1.7 \cdot 10^4$  rubidium-87 atoms is initially prepared in a quasi-1D harmonic trap with longitudinal and transverse frequencies  $\omega/2\pi = 5.4 \text{ Hz}$  and  $\omega_\perp/2\pi = 70 \text{ Hz}$ , respectively. In the following we adopt the harmonic oscillator units:  $E_0 = \hbar\omega$ ,  $l_0 = \sqrt{\hbar/m\omega}$  and  $t_0 = 1/\omega$  for energy, length and time, respectively. When the trapping potential is turned off and the speckle potential is turned on the expansion of the atomic cloud is initially dominated by the particle interactions until the density drops significantly and the atoms start feeling only the disorder potential. This initial stage of the gas expansion sets the momentum distribution of the atoms which may be approximated by an inverted parabola with an upper cut-off  $k_{max} = 2\sqrt{\mu} = 12.7$  where  $\mu$  is the initial chemical potential of the system [23, 26].

The disorder potentials we choose are attainable in the

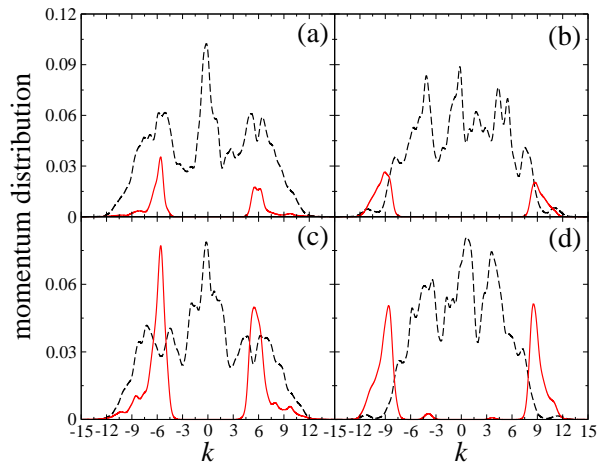


Figure 2: (Color online) Momentum distributions of the atoms localized in the disorder potential (dashed black lines) and the atoms which escaped from the disorder area (solid red lines). Panels (a) and (c) show the results for a single experimental realization of the disorder with parameters as in Fig. 1a,c while panels (b) and (d) are related to the parameters as in Fig. 1b,d. Panel (a) corresponds to the evolution time  $t = 100$  (2.9 s), panel (b) to  $t = 70$  (2 s), panels (c) and (d) to  $t = 200$  (5.7 s). Fraction of atoms that escaped the disorder region is about 9% in (a) and (b), and 20% in (c) and (d). In order to take into account experimental resolution all data have been convoluted with Gaussian of  $\Delta k = 0.3$  width. All values are presented in the harmonic oscillator units, i.e. energy  $E_0 = \hbar\omega$ , length  $l_0 = \sqrt{\hbar/m\omega}$  and time  $t_0 = 1/\omega$  where  $\omega/2\pi = 5.4$  Hz.

experiment reported in Ref. [21], i.e. they extend 862 units (4 mm) along the  $z$  direction with the correlation length  $\sigma_R = 0.066$  (0.31  $\mu\text{m}$ ). We consider the potentials obtained by introducing one or two obstacles in the diffusive plate that are presented in Fig. 1. If the atomic cloud starts at the center of the disorder potentials and if the cut-off of the momentum distribution is  $k_{max} \lesssim 13$  then we may expect that time evolution leads to emission of atoms with  $|k| \approx 5.5$  in the single obstacle case (cf. Fig. 1a,c) and with  $|k| \approx 9$  in the case of two obstacles (cf. Fig. 1b,d). In the latter case we may expect also a small leakage of atoms with  $|k| \approx 3.5$ . For  $|k| \approx 3.5$  the localization length shown in Fig 1d reaches locally a maximum value of  $l_{loc} \approx 120$ . Before the particle interactions become negligible the gas spreads over a significant range of the disorder region. Therefore the Anderson localization is not able to diminish completely the leakage of atoms with  $|k| \approx 3.5$ .

Starting with the ground state of the stationary Gross-Pitaevskii equation [9–15] in the presence of the harmonic trap, we integrate the time-dependent Gross-Pitaevskii equation when the trap is turned off and a disorder is turned on. Figure 2 shows momentum distributions of atoms that escaped from the disorder region and those that are localized for the disorder potentials correspond-

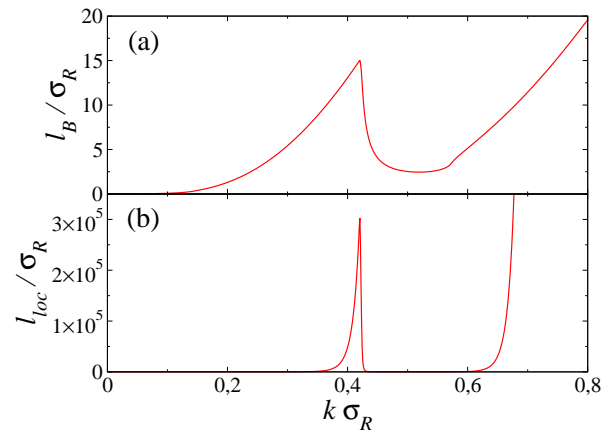


Figure 3: (Color online) The Boltzmann transport mean-free path (a) and the localization length (b) for atoms in the 2D speckle potential created by transmission of a laser beam through the circularly shaped diffusive plate with the obstacle in the form of a ring, i.e. the aperture of the optics is described by  $\Theta(R - |\mathbf{r}|) - \Theta(\rho - |\mathbf{r}|) + \Theta(\zeta - |\mathbf{r}|)$  with  $\sigma_R/\sigma_\rho = 0.99$  and  $\sigma_R/\sigma_\zeta = 0.15$ . The potential strength corresponds to  $\eta = 0.15$ . The quantities presented in the figure are dimensionless.

ing to Fig. 1 at different moments in time. The expected selective spectral emissions of atoms are apparent in the figure. Interestingly in Fig. 2d, i.e. for longer evolution time, small peaks around  $|k| \approx 3.5$  become visible.

Finally let us consider a possibility of the realization of an atom analog of an optical random laser in 2D. The Boltzmann transport mean-free path  $l_B$  is the characteristic spatial scale beyond which memory of the initial direction of the particle momentum is lost. In 2D  $l_{loc} = l_B e^{\pi k l_B / 2}$  and thus the localization length is much larger than  $l_B$  which is in contrast to the 1D case where these two quantities are nearly identical  $l_{loc} = 2l_B$  [19]. For the circularly shaped diffusing plate with the radius  $R$  the classical transport mean-free path [25, 26], to the second order in the potential strength, reads

$$\frac{1}{l_B} = \frac{\eta^2}{k\sigma_R^2} \int \frac{d\phi}{2\pi} (1 - \cos \phi) \mathcal{P} \left( 2k\sigma_R \sin \frac{\phi}{2} \right), \quad (4)$$

where  $\eta = V_0/E_\sigma$  is the ratio of the potential strength and correlation energy  $E_\sigma = \hbar^2/(m\sigma_R^2)$  with  $\sigma_R = \alpha/R$ . The power spectrum  $\mathcal{P}(k)$  of the optical speckle potential disappears for  $k \geq 2/\sigma_R$ . Nevertheless, the  $l_B$  (and consequently also  $l_{loc}$ ) is always finite. In the bulk 2D system, an initially prepared atomic wave-packet follows a diffusive motion at short time but eventually the dynamics slow down and freeze due to the Anderson localization process [25–27].

By introducing obstacles in the diffusive plate we are able to shape the power spectrum of the speckle potential. On one hand the fact that  $\mathcal{P}(k)$  may disappear at certain momentum intervals does not mean divergence of the corresponding transport mean-free path (4). On

the other hand any non-monotonic behaviour of  $l_B(k)$  is dramatically amplified in the behaviour of  $l_{loc}(k)$  because the localization length is an exponential function of  $l_B$ . In Fig. 3 we present an example related to the obstacle in the form of a ring, i.e. the aperture of the optics is described by  $\mathcal{A}(\mathbf{r}) = \Theta(R - |\mathbf{r}|) - \Theta(\rho - |\mathbf{r}|) + \Theta(\zeta - |\mathbf{r}|)$ . At  $k\sigma_R \approx 0.4$  both  $l_B$  and  $l_{loc}$  shows a maximum. However, while the transport mean-free path changes by only a few in the neighboring region the localization length changes by four orders of magnitude. If the width of the momentum distribution of a BEC loaded in such a disorder potential is smaller than  $0.6/\sigma_R$  and the radius of the disorder medium is greater than  $10^3\sigma_R$  but less than  $10^5\sigma_R$  the multiple scattering process leads to an isotropic emission of atoms with  $k \approx 0.4/\sigma_R$ .

We have outlined a proposal for the realization of a matter-wave analog of an optical random laser. Spatially correlated disorder potential for atoms with a peculiar pair correlation function can be created by transmitting a laser beam through a diffusive plate with obstacles. The resulting Anderson localization length reveals non-monotonic behaviour as a function of particle momentum. It allows for filtering momenta of particles that leave the area of the disorder, if the size of the disorder medium is suitably chosen. The disorder medium is assumed to be initially loaded with a BEC which guarantees that the matter-waves emitted from the medium are coherent. We have restricted ourselves to the 1D and 2D cases but the atom analog of an optical random laser can be also anticipated in 3D. In 3D the Ioffe-Regel criterion discriminates between waves that are Anderson localized ( $kl_B \lesssim 1$ ) or not [19, 28]. Thus, a spatially correlated disorder potential for which the Ioffe-Regel criterion is not fulfilled for specific momenta should allow for selective emission of matter-waves in 3D.

Our proposal is directly applicable to atomic matter-wave experiments. From the point of view of the optical random lasers our analysis is not complete because it is restricted to passive random materials without gain. There is an interesting question whether a disorder with properties similar to those analyzed here play a role in optical random lasers and which modes are important when the gain is included in a system.

The non-monotonic behaviour of the localization length results in the appearance of a multiple effective mobility edge if a disorder system is finite. Wave transport is then unusual and interesting on its own. A shallow non-monotonical behaviour of the Anderson localization length versus energy has been observed also in a classical wave system, see Ref. [29].

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*Note added:* After submission of this article, we became aware of a related theoretical study [30].

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- [1] D. S. Wiersma, *Nature Physics* **4**, 359 (2008).
  - [2] V. S. Letokhov, *Zh. Eksp. Teor. Fiz.* **53**, 1442 (1967); *Sov. Phys. JETP* **26**, 835 (1968).
  - [3] N. M. Lawandy *et al.*, *Nature* **368**, 436 (1994).
  - [4] D. S. Wiersma and A. Lagendijk, *Phys. Rev. E* **54**, 4256 (1996).
  - [5] H. Cao *et al.*, *Appl. Phys. Lett.* **73**, 3656 (1998); H. Cao *et al.*, *Phys. Rev. Lett.* **82**, 2278 (1999).
  - [6] S.V. Frolov *et al.*, *Phys. Rev. B* **57**, 9141 (1999); T.V. Shahbazyan *et al.*, *Phys. Rev. B* **61**, 13 266 (2000).
  - [7] D. S. Wiersma, *Nature* **406**, 135 (2000).
  - [8] M. H. Anderson *et al.*, *Science* **269**, 198 (1995).
  - [9] M.-O. Mewes *et al.*, *Phys. Rev. Lett.* **78**, 582 (1997).
  - [10] E. W. Hagley *et al.*, *Science* **283**, 1706 (1999).
  - [11] I. Bloch *et al.*, *Phys. Rev. Lett.* **82**, 3008 (1999).
  - [12] G. Cennini *et al.*, *Phys. Rev. Lett.* **91**, 240408 (2003)
  - [13] W. Guerin *et al.*, *Phys. Rev. Lett.* **97**, 200402 (2006)
  - [14] A. Couvert *et al.*, *Europhys. Lett.* **83**, 50001 (2008).
  - [15] J. T. Schulte *et al.*, *Phys. Rev. Lett.* **95**, 170411 (2005).
  - [16] D. Cl  ment *et al.*, *New J. Phys.* **8**, 166 (2006).
  - [17] P.W. Anderson, *Phys. Rev.* **109**, 1492 (1958).
  - [18] P. A. Lee and T.V. Ramakrishnan, *Rev. Mod. Phys.* **57**, 287 (1985).
  - [19] B. van Tiggelen, in *Diffuse Waves in Complex Media*, edited by J.-P. Fouque, NATO Advanced Study Institutes, Ser. C, Vol. 531 (Kluwer, Dordrecht, 1999).
  - [20] I. M. Lifshits, S. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems* (Wiley, New York, 1988).
  - [21] J. Billy *et al.*, *Nature* **453**, 891 (2008).
  - [22] G. Roati *et al.*, *Nature* **453**, 895 (2008).
  - [23] L. Sanchez-Palencia *et al.*, *Phys. Rev. Lett.* **98**, 210401 (2007)
  - [24] E. Gurevich and O. Kenneth, *Phys. Rev. A* **79**, 063617 (2009); P. Lugan *et al.*, *Phys. Rev. A* **80**, 023605 (2009).
  - [25] R. C. Kuhn *et al.*, *Phys. Rev. Lett.* **95**, 250403 (2005)
  - [26] C. Miniatura *et al.*, *Eur. Phys. J. B* **68**, 353 (2009).
  - [27] M. Robert-de-Saint-Vincent *et al.*, *Phys. Rev. Lett.* **104**, 220602 (2010).
  - [28] R. C. Kuhn *et al.*, *New J. Phys.* **9**, 161 (2007).
  - [29] D.   peta *et al.*, *Phys. Rev. A* **84**, 011801(R) (2011).
  - [30] M. Piraud, A. Aspect, L. Sanchez-Palencia, arXiv:1104.2314.